Economic theory shows how markets work, through the determination of prices and quantities. In a world of certainty, decisions involve predictable streams of costs and benefits. Converting these into present value one can compare them and choose the optimal decision (that is, the one which maximises the benefits, net of costs).

An optimal decision must be characterised by a positive present value (PV):

\[ PV = \sum_i \frac{(B_i-C_i)}{(1+r)^i} = \sum_i \rho^i P_i > 0 \]

where \( P_i = B_i - C_i \) denotes the net benefits (profits, given by the difference between benefits and costs), \( r \) is the rate of discount, \( \rho^i = \frac{1}{(1+r)^i} \) the discount factor and \( i=1,\ldots \) refers to time.

An optimal decision must be also characterised by the highest values among all possible alternatives. Otherwise, undertaking such a decision would imply giving up an alternative one with a higher (positive) present value.

In an uncertain world the analysis of individual choices is more complex. However, under some simplifying assumptions we can convert the problem in the already solved problem. Specifically, we can assume that each individual has a probability distribution over possible outcomes. He does not know what will happen, but he knows the likelihood with which each outcome will be realised. His problem is how to maximise his welfare. He can choose the alternative that gives the highest expected return (or utility), i.e. the sum of returns (utility) associated with the different possible outcomes, each weighted by its probability. In what follows, we analyse the implications of uncertainty, considering the utility flow from a year’s expenditure, while temporarily ignoring other complications. To make things simple, we talk in “euros” and “utiles” instead of “euros per year” and “utiles per year”, i.e. an income of x euros/year for one year equals x euros.

### A. The expected return: Heads or tails?

Consider the case in which you are betting on whether a coin will come up heads or tails. You have 1 € and you can choose between a certain outcome (i.e. to decline the bet, ending up with 1 €) and an uncertain outcome (i.e. to accept the bet ending up with either more or less than 1 €). Using a fair coin, half the time it will come up heads. A rational gambler will take bets that offer a payoff of more than 1 € and refuse any bet that offers less. For instance, if he is paid 2 € when the coin comes up heads and he pays 1 € if it comes up tails, then, by accepting the bet, on average he gains 0.50 €. If he is offered 0.50 € for the risk of 1 €, then on average he loses 0.25 €.
by accepting the bet and should hence refuse the bet.

Taking the same gamble many times a gambler should choose the one with the highest expected return. He should take any bet that is better than a fair gamble i.e. one with positive expected return. The case of a gambler betting many times on the toss of a coin can be generalised to describe any game of chance, following the rule “to maximise expected return”. The expected return (E R) is the sum, over all of the possible outcomes, of the return from each outcome times the probability of that outcome.

E R  = \[ \sum \pi_i \cdot R_i \] with \[ \sum \pi_i = 1 \]

Here \( \pi_i \) is the probability of outcome \( i \) occurring, \( R_i \) is the return from outcome number \( i \).

Any gamble ends up with one of the alternative outcomes happening; for instance, when you toss a coin, it must come up either heads or tails. In this gamble, using a fair coin \( \pi_1 = \pi_2 = 0.5 \) are respectively the probabilities associated to the outcome heads and tails, according to which the gambler respectively gains \( R_1 = 2 \) € and loses \( R_2 = -1 \) €. The expected return is \( 0.50 \) €.

E R  = \( (\pi_1 \cdot R_1) + (\pi_2 \cdot R_2) = [0.5 \cdot (+2 \) €] + [0.5 \cdot (-1 \) € ]\) = \( + 0.50 \) €.

If you play the game many times, you will on average make \( 0.50 \) each time you play. The expected return from taking the gamble is positive, so you should take it, provided that you can repeat it many times. The same applies to any other gamble with a positive expected return. A gamble with a zero expected return is a fair gamble.

**Problem 1**: Check the result for \( \pi_1 = \pi_2 = 0.5, R_1 = 0.5 \) €, \( R_2 = -1 \) € by redoing the calculations. What happens if \( \pi_1 = 0.9 \) and \( \pi_2 = 0.1 \)?

Suppose now that you are playing the game once and that the bet is \( 50,000 \) €, i.e. all your income. If you lose, you starve, if you win, you gain only a modest welfare increase. You may feel that a decline in your wealth from \( 50,000 \) to zero hurts you more than an increase from \( 50,000 \) to \( 150,000 \) would help you. The euros that raise your income from zero to \( 50,000 \) are worth more (per unit) than the additional 100,000 euros, starting from an income equal to \( 50,000 \) €. The rule “to maximise expected return” is no longer rational. What is the rational behaviour in such a case?

**B. The expected utility: risk aversion and risk preference.**

John Von Neumann, the inventor of game theory, provided the answer to the question at the end of last section, by combining the idea of expected return used in the mathematical theory of gambling (probability theory) with the idea of utility used in
economics. In this way, he shows that it is possible to describe the behaviour of individuals dealing with uncertain situations.

The basic underlying idea is that instead of maximizing expected return in euros, individuals maximize expected return in utils, i.e. expected utility. Each outcome $i$ has an associated utility $U_i$. He defined expected utility as:

$$E U(R) = \sum_i \pi_i U(R_i)$$

The utility you get from outcome $i$ depends only on how much more (or less) money that outcome gives you. If utility increases linearly with income $U(R) = a + (b \cdot R)$, as shown along $OE$ on Figure 1, whatever decision maximizes $E R$ maximizes $E U$.

$$E U(R) = \sum_i \pi_i (a + b \cdot R_i) = a \sum_i \pi_i + b \sum_i \pi_i R_i = a + b \cdot E R$$

Hence, with a linear utility function the individual maximizing his expected utility behaves like the gambler maximizing his expected return.

We can represent graphically the utility level of any outcome on a bidimensional graph, such as $ODE$ in Figure 1. Along this curve we will then find the utility of the income $R^i$ associated with outcome $i$. Considering $ODE$ in Fig. 1, if you start with $R^* = 50,000 \, \text{€}$ and bet all of it at even odds on the toss of a coin (heads you win, tails you lose) then the utility to you of the outcome “heads” is the utility of € 1,300 (point $E$). The utility to you of the outcome “tails” is the utility of zero dollars (point $O$).

![Fig. 1](image-url)
ODE shows a relation where income has declining marginal utility. That is, total utility increases with income, but it increases more and more slowly as income gets higher and higher. In deciding whether to bet € 25,000, you are choosing between two different gambles. If you do not take the bet, you have the certainty ($\pi^* = 1$) of ending up with $R^* = € 50,000$. If you do take the bet, you have a 0.5 chance of ending up with $R^A = € 25,000$ and a 0.5 chance of ending up with $R^B = € 75,000$. So in the first case, assuming $U(50,000 \text{ €}) = 1,000$ utiles, we have:

$$E U(R^*) = \sum_i \pi_i U_i = \pi^* \cdot U^* = 1,000 \text{ utiles}$$

In the second case, with $U(25,000 \text{ €}) = 600$ and $U(100,000 \text{ €}) = 1,200$ we have:

$$E U(R) = \sum_i \pi_i U_i = (0.5 \cdot 600 \text{ utiles}) + (0.5 \cdot 1,200 \text{ utiles}) = 900 \text{ utiles}$$

The individual, taking the alternative with the higher expected utility, declines the bet. In money terms, the two alternatives are equally attractive; they yield the same expected return $R^* = 50,000$ €. In that sense, it is a fair bet. In utility terms, the sure option $U(R^*)$ is superior to the risky one $U(R^C)$. As long as the utility function has the shape shown in Figure 1, a certainty of $X$ € will be always preferred to a gamble with the same expected return $X$ €.

An individual who behaves in that way is defined as risk averse. Such an individual would decline a fair gamble but might accept one that is better than fair, i.e. bet € 1,000 against € 1,500 on the flip of a coin, for example.

ODB in Figure 2 shows instead the utility function of a risk lover. It exhibits increasing marginal utility. A risk lover is willing to take a gamble slightly worse than fair, although he would decline one with very low expected return. An individual who is neither a risk lover nor a risk averter is called risk neutral. Its corresponding utility function (line OE) is also shown in Figure 1.

**Problem 2:** Will the agent accept the bet for $U^* = 1,000$, $U^A = 200$ and $U^B = 1,800$?
with $\pi_A=\pi_B=0.5$ in figure 2? What are the values of $U^*$, $U^A$ and $U^B$ for a risk neutral agent in Figure 1? Will he prefer the certain income $R^*$ for $\pi_A=0.6$? And for $\pi_A=0.4$?

The degree to which someone exhibits risk preference or risk aversion depends on the shape of the utility function, the initial income, and the size of the bet. For small bets, we can expect everyone to be roughly risk neutral; the marginal utility of an euro does not change very much between an income of 49,999 € and an income of 50,001 €, which is the relevant consideration for someone with 50,000 € who is considering a 1 € bet.

C. Rational behaviour in an uncertain world.

As section B shown, it is relatively easy to predict the behaviour of someone maximizing expected return than of someone maximizing expected utility. Each individual can still maximise his utility by maximizing his expected return, as long as he can repeat the same gamble many times (so that he can expect results to average out). His income in the long run is (almost) certain. He maximises his expected utility making that income as large as possible, i.e. by choosing the gamble with the highest expected return. Maximizing expected utility is also equivalent to maximizing expected return (like the gambler we started with) when the: (i) individual is risk-neutral, (ii) the size of the prospective gains and losses is small compared to one’s income (we can treat the marginal utility of income as constant and changes in utility as proportional to changes in income, so he should act as if he were risk neutral).

Let us now consider a firm rather than an individual. The management, wishing to raise the present price of firms’ stock, maximises the expected value of its future price by maximizing the expected value of future profits. The threat of takeover bids forces management to maximise the value of firms’ stock. When management pursues his own goals the conclusion no longer holds. If the firm goes bankrupt, the income of the chief executive may fall a lot. Accordingly, he may not be willing to take a 50 percent chance of leading to bankruptcy even if it also has a 50 percent chance of tripling the firm’s value. Hence the assumption of risk neutrality might not be always appropriate for firms as well.

The existence of risk adverse agents explains the need for insurance. Suppose Paula’s income is 30,000 € and there is a small probability (0.01) that an accident reduces it to € 10,000. The insurance company offers to insure her against that accident for a fixed price of € 200. Whether or not the accident happens, she gives them € 200. If the accident happens, they give her back € 20,000. She has a choice between two gambles: to buy or not to buy the insurance. Buying the insurance, whether or not the accident occurs she has € 30,000 minus the € 200 paid for the insurance. For the first
gamble: $\pi_1 = 1$; $R_1 = 29,800 \, €$ and

$$EU = \pi_1 \cdot U(R_1) = 998 \, \text{utiles.}$$

When she does not buy the insurance: $\pi_1 = 0.99$; $R_1 = € 30,000$; $U(R_1) = 1,000 \, \text{utiles}$ and $\pi_2 = 0.01$; $R_2 = € 10,000$; $U(R_2) = 600 \, \text{utiles}$. That implies:

$$E U(R) = [\pi_1 \cdot U(R_1)] + [\pi_2 \cdot U(R_2)] = 990 \, \text{utiles} + 6 \, \text{utiles} = 996 \, \text{utiles.}$$

Since Paula is better off with the insurance than without it and will buy it. Notice that for $R_1 = € 30,000$ the marginal utility of $100€$ is about 1 utile.

**Problem 3**: How much would the agent in figure 1 pay to have a certain income $R^* = € 50,000$, instead of $R^A = € 25,000$ and $R^B = € 75,000$ with $\pi_1 = \pi_2 = 0.5$. How much should instead the agent in figure 2 be paid to have the certain income $R^*$?

Buying insurance was a fair gamble: € 200 are paid in exchange for 1% chance of receiving € 20,000. An insurance company making 100,000 bets will end up receiving, on average, almost exactly the expected return, if the probabilities of these bets are independently distributed. When insurance is fair, the insurance company and the client breaks even in monetary terms, but the client gains in utility.

In the real world, insurance companies incur additional expenses other than paying out claims and offer gambles somewhat less than fair to clients. Sufficiently risk averse consumers still accept the gamble and buy an insurance contract that lowers their expected return but increase their expected utility. In our case, with a marginal utility of $100€ \approx 1$ utile, it would still be worth buying the insurance even if the company charged € 300 for it. It would no longer worth buying at € 500.

**Problem 4**: Check those results.

Buying a lottery ticket is the opposite of buying insurance. When you buy a lottery ticket, you accept an unfair gamble but this time you do it in order to increase your uncertainty. In fact, on average, a lottery pays out in prizes less than it takes in. If you are risk averse, it may make sense for you to buy insurance, but you should never buy lottery tickets. If you are a risk lover it may make sense for you to buy a lottery ticket, but you should never buy insurance.

This brings us to the lottery-insurance paradox. In the real world, the same people sometimes buy both insurance and lottery tickets. They both gamble and buy insurance knowing the odds are against them. Is this consistent with rational behaviour? We propose two possible explanations. The individual is risk averse for one range of incomes and risk preferring for another, higher, range. What individuals get is not just one chance in a billion of a 100,000 € prize, but also, for a while, the daydream (at a very low price) of getting it, because they have a slim chance to
D. Von Neumann utility function and social welfare.

Von Neumann proved that if individual choice under uncertainty meets a few consistency conditions, it is always possible to assign utilities to outcomes in such a way that the decisions actually made follow from maximizing expected utility.

He considers an individual behaviour “rational” or “consistent” under uncertainty (i.e. choosing among “lotteries”: a collection of outcomes, each with a probability) if: (i) given any two lotteries A and B, the individual either prefers A to B, prefers B to A, or is indifferent between them, (ii) preferences are transitive; if you prefer A to B and B to C, you must prefer A to C, (iii) in considering lotteries whose payoffs are themselves lotteries people combine probabilities in a mathematically correct fashion, (iv) preferences are continuous, (v) when outcome A is preferred to outcome B and outcome B to outcome C, there is a probability mix of A and C (a lottery containing only those outcomes) equivalent to B; i.e. since U(A) > U(B) > U(C) as utility moves from U(A) to U(C), at some point, it must be equal to U(B).

Accepting these axioms and hence the Von Neumann utility the statement “I prefer outcome X to outcome Y twice as much as I prefer Y to Z” is equivalent to “I am indifferent between a certainty of Y and a lottery that gives me a two-thirds chance of Z and a one-third chance of X”.

Problem 5*: Check that these statements are equivalent by doing the calculations.

We can make quantitative comparisons of utility differences and quantitative comparisons of marginal utilities. The principle of declining marginal utility is equivalent to risk aversion. We agree about the order of preferences and about their relative intensity, but we may still disagree about the zero of the utility function and the size of the unit in which we are measuring them. Utility functions are arbitrary with respect to linear transformations. Changes in the utility function that consist of adding the same amount to all utilities (changing the zero), or multiplying all utilities by the same number (changing the scale), or both, do not really change the utility function, i.e. the behaviour is exactly the same. [Please check this statement].

Utilitarian used the concept of utility, to determine social welfare, i.e. the total utility of individuals that society should maximise. This was criticized because there is no way of making interpersonal comparisons of utility, nor of deciding if a change that benefits me and hurts you increases total utility. With Von Neumann utility the utilitarian rule “to maximise total utility” is equivalent to “choose the alternative you prefer if you were one of the people affected”. Given the probability $\pi = 1/N$ of being
anyone; if there are N individuals, we can write the utility of person i as \( U_i = U \) and consider the expected utility of the lottery with probability \( \pi \) of being each person:

\[
E U(R) = \sum_i \pi_i U_i = \sum_i \pi U_i = \pi \sum_i U.
\]

It is easy to see how \( \sum_i U_i \) is simply social welfare, i.e. society’s total utility.

**Problem 6:** Would this result hold when individual utility functions are different?

**Appendix: Solution to selected problems**

**Problem 1:** Check the result for \( \pi_1 = \pi_2 = 0.5 \), \( R_1 = 0.5 \) \( € \), \( R_2 = -1 \) \( € \) by redoing the calculations. What happens if \( \pi_1 = 0.9 \) and \( \pi_2 = 0.1 \)?

\[
E R = (\pi_1 \cdot R_1) + (\pi_2 \cdot R_2) = [0.5 \cdot (0.5 \) \( €)] + [0.5 \cdot (-1 \) \( €)] = -0.50 \) \( € \).
\]

The expected return from taking the gamble is negative.

It becomes positive if \( \pi_1 = 0.9 \) and \( \pi_2 = 0.1 \).

\[
E R = (\pi_1 \cdot R_1) + (\pi_2 \cdot R_2) = [0.9 \cdot (0.5 \) \( €)] + [0.1 \cdot (-1 \) \( €)] = 0.35 \) \( € \).
\]

**Problem 2:** Will the agent accept the bet for \( U^* = 1,000 \), \( U^A = 200 \) and \( U^B = 1,800 \) with \( \pi_A=\pi_B=0.5 \) in figure 2? What are the values of \( U^* \), \( U^A \) and \( U^B \) for a risk neutral agent in Figure 1? Will he prefer the certain income \( R^* \) for \( \pi_A=0.6 \)? And for \( \pi_A=0.4 \)?

Yes, being indifferent;

\( U^* = 1,000 \), \( U^A = 500 \) and \( U^B = 1,500 \);

No.

**Problem 3:** How much would the agent in figure 1 pay to have a certain income \( R^* = € 50,000 \), instead of \( R^A = € 25,000 \) and \( R^B = € 75,000 \) with \( \pi_1=\pi_2 = 0.5 \). How much should instead the agent in figure 2 be paid to have the certain income \( R^* \)?

In both case it is the segment \( B^*B^C \)

**Problem 4:** Check those results.

\( U(29,700)= 997 \) and \( U(29,600)= 995 \)

**Problem 5:** Check that these statements are equivalent by doing the calculations.

Lottery 1 consist of Y, Lottery 2 of a \( 2/3 \) chance of Z and a \( 1/3 \) chance of X. Rearranging statement 1 we find it implies statement 2

**Problem 6:** Would this result hold when individual utility functions are different?

Yes, in fact: \( E U(R) = \sum_i \pi_i U_i(R_i) = \pi \sum_i U_i(R_i) \)