VON NEUMANN'S GROWTH MODEL
AND PUBLIC SECTOR ECONOMICS

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The main aim of a number of works has been to expand the original von Neumann theory of economic growth. Von Neumann's work has been interpreted and extended by Champenowne, Kemeny, Morgenstern, Thompson, Haga-Otsuki, Morishima, Fisher and many other economists with varying degrees of success and in a variety of directions, one feels that these and further developments are still possible.

The following are the main simplified assumptions of the von Neumann's model of an expanding economy.
1) Every good is involved in every process of production,
2) there are no workers' savings or capitalists' consumption,
3) consumer choice is not allowed for,
4) wages are predetermined (at subsistence level),
5) perfect competition is assumed,
6) natural factors of production are available in unlimited amounts,
7) there is no public sector.

Many have been examined and replaced by more general or more appropriate ones. Rarely however it has been taken into account the presence of externalities and of a public sector which produces and consumes.

It may be useful first to summarize the work already done in order to point out possible future perspectives for research.

It is particularly helpful to divide the simplification present in the von Neumann's assumptions into two main categories:

a) wages are predetermined, workers do not save, capitalists do not consume and consumer choice is not allowed for,
b) perfect competition is assumed, in the absence of any type of externalities and public goods, and there is no public sector or any kind of public intervention.

The previous interpretations and contributions represent extensions of
von Neumann's model, almost exclusively oriented towards incorporating additional consumption by non-production sectors (such as families). In relation to assumptions (a) the work started with Champenowne, Kemeny, Morgenstern, Thompson was virtually completed with Haga-Otsuki's and Morishima's contributions. In fact Morishima incorporates proper consumption demand and primary factor supply functions into the von Neumann economy.

On the other hand, for a long time less attention has been devoted to point (b) and no major progress has been achieved towards an economically meaningful introduction of the public sector in the consumption and production side of the original model. Rather, externalities and public commodities have not been satisfactorily dealt with as market failure and in connection with public intervention. After a critical examination of the existing contributions it is natural to try to generalize and integrate them into comprehensive models, to deal with different cases and compare possible balanced growth equilibria in order to examine the possible influence of the public sector. That has been partially done by Vagliaisindi who introduces each externality as a different von Neumann commodity, distinguished on the basis of its user. This approach (which is more than an useful devise) may be seen as a direct development of Samuelson's idea of individual consumption of public good, which has been criticized because it is not observable.  

On the problem of public inputs we have a recent specialistic literature in the area of public finance, but perhaps the first hint is already contained in the analogy between joint supply and collective consumption good, pointed out by Samuelson.  

Many writers, like Katzuca K. (1965), Sandmo (1972) and Boadway R. (1973), derived efficiency conditions for the supply of public intermediate commodities which are similar to Samuelson's summation rule (for collective consumption). But later Henderson (1974) examines a case under which the "Samuelson summation rule" is no longer meaningful. In his second special case in equilibrium an industry benefiting from public intermediate goods will consist of a large number of firms of infinitesimal size. The distinction

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1 In this context lies the main contribution of Vagliaisindi (1988).

between "factor augmenting" and "firm augmenting" (which depend on the fact that the 'publicness' is at the firm or at the industry level) is later proposed by McMillan (1979) and is similar to Meade's original one (between "unpaid factor" and "creation of atmosphere" type of externality)\(^3\) applied at the firms' level. The "Samuelson summation rule" fails in the case of "firm-augmenting public goods".

This specialistic literature usually assumes smooth production functions and examines partial equilibrium situations. Consequently, it seems quite interesting to deal with these problems in a general equilibrium growth setting, using von Neumann production functions.

That is true from at least two possible perspectives. First, this approach may allow more flexibility in empirical application and the use of linear programming methods.

Second, von Neumann production functions with joint products permit the economic lifetime of capital goods to be determined endogenously and achieve a clear distinction between public services (of a one period duration) and public assets (which last for more than one period). These are useful contributions that develop further the usual public finance treatment of the subject. Indeed they enable us to take into account problems which are quite relevant from a theoretical and practical point of view.

In fact it is easy to distinguish not only between a public consumption good and a public intermediate commodities but also between a public intermediate services and the asset that provides these services through it existence or through a more or less complicated production process. Two types of public goods that has most often being ignored, until Kalzuca.

Thus referring to the production side we may propose to name

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\(^3\) The essential difference between the two seems to lie in that the effects of external economies on a given industry do or do not depend upon its scale of operation. In the "creation of atmosphere" case the conditions which affect the output of a given industry are independent from its production scale. In this case "there are constant return in each industry to those factors which it controls and pays for, but ... there are not constant returns for the two industries taken together, the scale of operations being important in the one industry because of the atmosphere which it creates for the other." In the second (unpaid-factor) case "there are constant returns in society but not ... in each industry to the factors which each industry employs and pays for".
externality:

i) public intermediate commodities (pic), when only the production side is interested and the effects of this commodity last just for one period,

ii) public factor of production (pfp), if it is used in the production sector at least for two periods.

If one deals with public factor of production the one should enter into a multiperiod analysis and possibly into a growth setting. It is easy and natural to examine the case of a public capital good, proposed by Milleron, using von Neumann linear multi-output production function, or to deal with primary factors of production of the 'unpaid factor' like Kohli (1985). Vagliasindi (1988) case of public intermediate services is an approach to the problem of public intermediate goods, that may deal with both cases (i) and (ii).

However, before entering into these types of extensions it is important to re-examine thoroughly the possible interpretations of the original von Neumann model and to compare it with the original paper. 4

Then let us pin-point some interesting aspects and offer new views and interpretations about the original von Neumann's model.

One can argue in favour of the general thesis that the von Neumann's growth model is a real breakthrough in economics for more than the mathematical formalization alone. It has a lot of merit in the treatment of capital, in the rigourous economic solution of the problems of free goods and choice of techniques and in the interpretation of classical and neoclassical economic thought. Hence von Neumann is not a pure formalist but a first class mathematical economist not interested in axiomatization for its own sake.

For instance one may support this view, in relation to the original von Neumann theory of economic growth and its expansion remembering that:

- In von Neumann β is a monetary variable so that it would be correct to think of its equilibrium level as a sort of "natural interest rate".

- It is not completely true that the KMT assumptions are more general


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4 This work has in itself some original merit and has partially been done among others by Vagliasindi (1989). That paper examine some contributions unduly ignored in previous review articles and study all the extensions critically from an economic point of view.
than the von Neumann original ones. In fact the set of models which satisfy von Neumann's original assumptions is not contained in the set for which KMT assumptions are satisfied. Furthermore KMT assumptions may be too strict for an open economy.

The Morgenstern Thompson model of an open expanding economy is just a linear programming model of what we may call a constrained economy. It was originally achieved as a linear programming problem related to the von Neumann economy. Unfortunately the new axioms, derived in this way, do not have an immediately meaningful economic interpretation. Nevertheless, it may be useful as a first step towards an open economy and, above all, the treatment of public sector constraints on the production system.

However the Morgenstern Thompson's contribution (on an open expanding economy) has shown us how we may deal with an open economy in a closed model. This analytical definition is therefore too general and it should be clarified whether the von Neumann economy is closed in a stricter economic sense, following Georgeescu-Roegen, Koopmans and Samuelson. In the ambit of the von Neumann's model it is important to examine the exact meaning and relevance of different definitions of closed economy in connection with the treatment of labour.

In fact the problem is clearly connected with the rather special treatment reserved by von Neumann for labour, which was not accepted by those great economists, accustomed to treat labour as an ordinary good whose price is determined in the market. Von Neumann never assumes consumption as a labour producing process (which is an unsatisfactory hypothesis). Consequently he already deals with an open economy because labour is not produced by the system.

When one examines the different ways of introducing consumption in von Neumann's model, one should make a distinction between the models that introduce proper demand and supply functions and the ones which do not. Differently from previous interpretations (due to Frisch and Bauer) the Morgenstern-Thompson model belongs to the last category, even if we take into account their latest contribution of 1976, that is their additional income
matrix which should "lead to economically reasonable models".\(^5\)

In the very same paper Morgenstern and Thompson also tried to introduce a public sector as a new consumer and saver in the von Neumann model. It is possible to propose different economic interpretation of their model as well as a new original model; the "very simple public sector model", contained in Vagliasindi (1989). This one is consistent with the previous Morishima-Nosse econometric specification and, consequently, can represent in a more effective way the actual world.\(^6\)

However for a more general treatment of the public sector in the von Neumann model it is necessary to introduce externalities and public commodities. This has been done by Morishima-Thompson and attempted by Fisher in an activity analysis way.

Fisher (1977) explicitly introduced externalities and processes which produce both goods and bads. However his generalization is not completely free from criticism. Some shortcomings can be solved introducing each externality as a different commodity, distinguished on the basis of its user. But these public intermediate commodities (mainly related to the production side of the economy), which influence directly the use and the intensity of one or more productive processes, are also one of the premises and foundations of the public sector normative and positive theory of De Viti De Marco.

Consequently dealing with public intermediate commodities one automatically builds what can be called a De Viti De Marco - von Neumann model, in honour of this great public finance scholar. In this context Vagliasindi discusses two possible model with positive unpaid factor

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\(^5\) In this context one can support and generalize the interpretation hinted at by M. C. Lovell against Frisch and Bauer. That, however, does not affect the lack of economic analysis in Morgenstern-Thompson's aggregate demand and the peculiarity of the income matrix. In comparison the positive interest rate hypothesis and the introduction of the free good rule in the labour market by Haga-Otsuki are just minor slips.

\(^6\) In previous works on multisector economies as Petretto (1976) and Morgenstern and Thompson (1976), the externality problem were ignored.
externalities that obey to the "Samuelson summation rule".

It is interesting to notice how the very fact that public commodities are handled within the von Neumann model may also help to clarify the problems encountered by the "Samuelson summation rule" in the case of "firm-augmenting public goods". It should be noted that the same type of problem was met by Samuelson himself who was not completely satisfied of the analogy between joint supply and collective consumption good, because it may result in an incorrect hint. It is sometimes thought that increasing the number of citizens who are jointly supplied public goods leads to a similar determinate result. This is a reasoning from an incorrect analogy. 7

Thus, following Samuelson hint, the problem clearly emerges in the von Neumann context and depends on the things that are the atoms of the economy in each model (in our case industries and firms) for instance: (a) each economic agent able to run certain processes, (b) each production process of the economy.

Even the so called firm-augmenting case corresponds to case (a) if we assume that firms minimal size is at an individual level and we do not allow to increase the number of citizens who are jointly supplied public inputs. 8

In the case (b) when use of public inputs by distinct processes is non-rival, and that the quantities of pic required depend on the intensity of each process we have the factor augmenting case. However if we allow all the processes to be present twice (or more) in the input and output matrices the new economy will be able to obtain the same output of private goods with half (or less) of the previous input of public goods. Each two processes will require half of the previous amount of pic and the total demand will be halved because the use of pic is not rival between two identical processes. Keeping on duplicating the demand for pic will decrease tending to zero in accordance with Henderson second case.

Thus, in the ambit of a von Neumann model it seems easier to end misunderstandings (superfluous differentiations) and to re-adjust Samuelson's analysis to factors of production and intermediate commodities.

8 This has been hinted also in Vagliasindi (1988) when Fisher's model is examined.
It is quite surprising that this model, which incorporates the unpaid factor economies with public intermediate commodities, can be reduced to the original von Neumann economy. That is a strong support to our general thesis on von Neumann.

Thus, the very heart of the Italian doctrine of public finance can be reformulated. Taxes are levied on the production processes as the prices to be paid for the use of public commodities. In the private market equilibrium those pseudo-prices are no longer determined by the equilibrium conditions but are simply set equal to zero.

In this way it has been shown how the rate of growth of the von Neumann economy, where the prices of public commodities are determined by the equilibrium solution and paid through taxation, is greater than the one which would be determined by the private market equilibrium where public commodities are free.

This analysis shows the possible usefulness to identify the creation of each externality (intended in the broader sense) with the production of a distinct commodity dealing in this way with the limiting case of the public good itself. Furthermore it enables us to determine, by finding an equilibrium point, the choice between private and public services and the processes to produce them and the degree of public use of any given asset (when a public asset may produce private goods).

Vagliasindi (1991) consider the specific aspects of productive activity which give rise to adverse environmental effects of a local nature due to the "creation of atmosphere". Analyzing a model with bads, he introduces an alternative approach, which does not use smooth production functions, and examines a different externality opposed to the unpaid factor type assumed by Morishima and Thompson.

A new concept of equilibrium, the constrained von Neumann equilibrium (in which additional constraints, derived from the presence of bads, are allowed for) is introduced in order to justify the presence of a public sector and it is assumed that each industry has a sufficiently large number of basic processes in order to approximate effectively any smooth neoclassical production function (with decreasing returns) like the ones adopted by Morishima and Thompson (1960).
In order to find the equilibria of this model one can apply a modified version of the Morgenstern-Thompson linear programming model of a constrained economy. Hence it has been found an economic reason for the fact that in their model the public sector imposes minimum and maximum intensity levels to some of the processes of the economy.

In this case, however, the presence of a public sector increases the rate of growth and of interest. Furthermore, differently from previous models, there may not exist a private market equilibrium. All the equilibria can be achieved only by a mixed economy (through public intervention).

Let us give a closer look to the basic arguments. In the ambit of a linear production model the 'lethal vector' level of a process of production is the maximum level of bads at which that process is operational. Lethal vectors are intrinsic in the productive system while one or more community 'health constraints' (imposed from outside to the whole economy or local sub-economies) may be exogenously introduced by the government and local authority.

In general, each lethal vector is associated with one production process and determines if (given the present level of bads) it can be run or not. The economic agents consequently cannot run the initial processes when the lethal level constraint is not satisfied. This standard exists and individual agents are not free to consider it or not.

The hypothesis of a single lethal vector may be used as an heuristical devise which allows us to examine the simplest case in which it is clear that the private market equilibrium is not viable for the whole system. In fact, this additional constraint is not always satisfied by the von Neumann equilibrium of the entire economic system.

Introducing (along with the more profitable original processes) alternative processes which are not affected by the level of bads one may consider a sub-economy (composed by all the processes) associated with an infinite lethal vector of bads. Even if an equilibrium of such a sub-economy exists it does not always represent a private market equilibrium for the general economic system.

In order to demonstrate it, let us assume that for any price vector associated with this equilibrium there exists at least one profitable original process of an industry. However, this means that in such initial state private
agents will find profitable to run such a process at an intensity different from zero. Thus the sub-economy equilibrium could not represent a private economy equilibrium.

A private economy equilibrium will exist only in the case in which the general economic system casually satisfy the lethal vector constraint or the sub-economy does not so that no original process can be run (and will be admissible only if the health constraint is casually fulfilled too).

In general the previous sub-economy equilibrium will be an equilibrium for the general economic system only when private agents will not find profitable to run any other feasible process. This implies that the State should tax this processes in order to obtain a mixed economy equilibrium in correspondence to the sub-economy equilibrium.

But the government can do even better allowing to use some profitable original processes up to a certain point (till the intensity is not so large that the lethal constraint is not satisfied). This is the fundamental result implicit in the Morgenstern-Thompson constrained economy that we can use even for more complex cases.9 Clearly in this way the government influences the behavior of individual agents. In the first case the government makes unprofitable all the non-alternative production processes, while in the later case he is able (through a tax subsidy scheme) to regulate the intensity of all the processes in order to find the maximal rate of growth of the entire economy.

The models already proposed show the need of public intervention even on purely productive ground and of a general equilibrium analysis. In the ambit of the existing public finance literature they are 'anomalous' but they may represent a first step toward an extension of von Neumann's model which can more realistically scrutinize the substance and the relevant implications of externalities related to the production side of the economy. The high potentials of von Neumann's performance have not yet been fully explored and the public sector theory can be developed (analysing the production side and identifying the creation of each externality with the production of a distinct

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9 This is an obvious consequence of the minimizing and and maximizing linear problems. Cf; Morgenstern and Thompson (1976) Theorem 4-6 at page 75.
commodity) dealing with public commodities in multiperiod models with joint production (in order to allow for public assets).

In this context one may feel that there is still plenty of space for new useful and economically meaningful extensions of the von Neumann's model. The only real limits are given by the steady state equilibrium setting, not by the linear processes, as has been shown by the previous works. One interesting possibility is to integrate in the family and the public sector better, fully allowing for consumers' public goods. For instance the "very simple public sector model" may represent a first step in this direction. In any case even if one decides to abandon the steady state equilibrium, the von Neumann production functions remain a very useful instrument of analysis.

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