Abstract. In School Year 2005/2006, we extended, in Italy, a part of a research initiated by Ewa Swoboda in Poland. Our research used simple tools but early results indicate that certain geometric aspects may depend on factors such as competency, psychology and gender. In this paper we present a preliminary analysis of the protocols.

THEORETICAL FRAMEWORK.

This work was inspired by interesting research carried out by Swoboda. She puts forward an interpretation of children’s protocols, based on van Hiele’s theory (van Hiele, 1986), a theory studying geometrical thinking and understanding. This theory divides the educational processes in geometry into different levels. In particular, the analysis of three initial levels shows a very important common aspect: manipulation. Van Hiele also distinguishes between rigid or feeble structures. Regarding the last point, the Polish researcher notices:

“In his opinion feeble structures are worth noticing, they fill out the majority of our everyday life. They come from a non-verbal, intuitional way of thinking, but mathematical thinking is not superior to the intuitional one. Feeble structures may be a beginning of knowledge on a higher level of thinking where we may have something to do with, ex. rigid structures or still a feeble one” (Swoboda, 2005b).

Swoboda shows that feeble structures can be used to study and analyse an activity based on the creation of a floor. Feeble structures are very important for educational research in order to detect the child’s thought. They can reveal the process of early geometric knowledge appropriation. They are expressions of geometrical thinking that cannot be expressed by word, but only by graphical language. Feeble structures are characterized by presence of connections, rotations, parallel translations, symmetries, applied only locally in the drawing (Fig. 1.a). Rigid structures reveal the presence of a mental project using both geometric shapes or isometries or the sake of regularity (Fig. 1.b, c, d, e). Pupil could pass through feeble structures to rigid ones by awareness of the ‘regularity’ and of isometric transformations, using sight. Swoboda (2006) shows that there is a relation between the presence of rigid structures and school success.

With the tiles in Fig. 3, we give a sort of alphabet for a language that can be considered the first step of a future expression by words. Therefore, following Vygotsky (Vygotskji, 1992), we helped the coming into existence of geometrical concepts.
The analysis of the children’s protocols shows that “… activities such as these described here give the opportunity to bring out many intuitions that can be treated as a basis for developing not very simple geometrical notions” (Swoboda, 2005b).

In our paper we will now discuss some of these intuitions.

Firstly, there is the concept of space: it can be limited (Fig. 1.c, e) or unlimited (Fig. 1.d) and intra-figural or inter-figural (Fig. 1b, e).

“Greek thought … tried to escape from the unlimited, considered a form of imperfection. For Aristotle there is no space above the sky of fixed stars” (Speranza, 1997).

This is the same conception which leads some learners in our activity not to stick on tiles which would go over the edge of the sheet of paper. Other learners conceive of space as unlimited, and have thus ‘in act’ conception of infinity (Marchini, 2004).

Moreover colour gives us much more information whether the pupil’s attention is on local features (Fig. 2.a) or it concerns the drawing as a whole (Fig. 2.b).

The second intuition is connected with important concepts of area and plane. An actual floor is designed by covering a plane space without gaps and without superimposing tiles. The activity is used in school for introducing the concept of area. A child who leaves “space” between the sides of the tiles does not still have the idea of covering. Usually there is a privileged reference frame, namely, the edges of the sheet, which makes ‘horizontal’ and ‘vertical’ array more likely. The research (Rozek & Urbanska, 1998) has shown children have different levels of awareness of ‘horizontal’ and ‘vertical’ organisation. Some use only ‘horizontal’ lines, others only ‘vertical’ ones. They lack the idea of bi-dimensional distribution and the concept of array that underlies multiplication. When using tiles with lines which make creating some “whole” or patterns possible, children can fell a need to arrange tiles one close to another, sometimes also in row-column order (Fig. 2.a).

Thirdly, the regular size of the tiles and the drawing on they (Fig. 3) can focus the children’s attention to the connection in order to construct continuous patterns. The colouring of drawings confirms that continuity is present in pupils’ mind even if,
from the real gluing of tiles, the connection is not complete (Fig. 2.c).
Moreover the act of constructing a tessellation requires a long sequence of
elementary acts: observation, ordering, copying, and repeating. Swoboda (2005a)
shows that drawing a pattern is not a mere perceptual copying, but it is a deep
thinking process which involves body and gestures (Marchini & Vighi, 2005).

“Of the domains of knowledge where children must enter, geometry is the one needing
the fullest cognitive activity, as it requires gesture, language and looking. It requires the
child to construct, to reason and to see, each activity indissoluble from the others.”
(Duval, 2005).

Arzarello (2004) emphasises the role of the body movements and gestures in
learning. Gesture expressiveness can be considered a sort of language useful to
understand pupils’ thoughts taking in account of the poor language competencies of
children of these ages.

For other aspects of the theoretical framework of this research we refer to (Swoboda
2005b) and the references therein. Our research used exactly the same setting as
(Swoboda 2005b).

THE RESEARCH.

Our research used exactly the same setting as (Swoboda 2005b), in order to compare
Polish and Italian results, but we made some innovations regarding the treatment of
the children’s protocols. Moreover by quantitative analysis of protocols we point out
some aspects that we feel interesting.

The first phase of the research consists of manipulation activities. The 212 pupils (97
- last year Kindergarten, 68 - first year Primary School, 47 - second year Primary
School) worked singly in classroom environment. The task was:

“Create from these tiles as beautiful floor as possible” (Swoboda, 2005b).

We can discuss this requirement, since it seems too ambiguous, but it allows the
children express themselves in a good way. In other words, the children are
completely free to choose which, how, where, how many tiles, and how many times,
in order to obtain the most beautiful flooring that they can. Looking for their
intuitions, we are interested in some geometrical order; other more precise statement
of the task would be more difficult to be interpreted by these young pupils.

The tiles, proposed in Kuřina, 1995, are of the following kinds.

Fig. 3: Kuřina, 1995 tiles.

The task offers an approach very different from the customary one in the school, the
one introducing ‘standard’ geometric figures. This approach can be presented very
early and it is motivating and well accepted also by kindergarten pupils; the time that pupils spent in constructing their works supports this conclusion. Moreover it offers an occasion for free explorations of the space (the blank sheet of paper). The children degrees of freedom in expressing their own intuitions are ‘limited’ by the structure of the tools used, but as the protocols shown, they are free enough. Moreover the mind activity required to construct and to colour the drawing is, in our opinion, a suitable, right task of geometrical activity necessary to prepare the next more formal treatment of geometry.

The ‘floor’ consists of an A4 blank sheet (21 cm × 29.7 cm). Each side of the tiles, which are square, measures 2.5 cm. The tiles do not therefore fit the sides of the paper. The children were not given access to scissors, and therefore had to face problems regarding their conception of space (for an investigation of this issue using different tools, see Marchini, 2004). The problem was worsened because our 14,700 tiles were slightly irregular having been photocopied and cut up by hand.

The whole experimental activity took one school year (2005-2006). The activity of each child was video-recorded, in order to allow a deeper analysis subsequently. We plan to analyse later films of children working on the tasks, with the precise aim of identifying deep thought processes manifested by body and gestures.

Each protocol was photocopied, the pupil coloured his/her protocol and gave it a title. The introduction of colour and title for protocols is the main distinction in the methodology of the research respect to original Swoboda one (Swoboda, 2005b), but since it is a subsequent step, we obtain black and white protocols that can be compared with the Polish research.

The colour affords new information about pupil’s aims. The colour can be also a kind of language, therefore we asked children to colour the ‘paving’ tessellation, as this might reveal the criteria the learners base their design on. The colour and the title given to the protocol, takes place of semantics for the black and white drawing.

From now on, we refer to tiles by names given to them by pupils. From left to right, in Fig. 3, we have: flowery, branch, straight and swallows. These names in themselves show a naturalistic interpretation of the tiles, not a geometric one. The same is true of many titles of coloured protocols. Remark static and dynamic views of tiles (swallows as trajectories of the birds). The task was to pave a floor, but the pupils tended to see it as an opportunity for self expression. They produced gardens or flowers (girls) and streets or racing tracks (boys).

THE PROTOCOLS.

Choosing and gluing each tile carefully it is possible to ‘save’ 4 mm and to cover exactly 27.5 cm of the ‘long’ side of the sheet. Another way is to place tiles not contiguously, leaving small regular gaps between them. In this way, the number of tiles that can be glued onto the sheet without going over the edges is 88. A floor
layer, probably, would adopt this technique, with irregular tiles and in case of he has not the possibility of cutting them. We consider this number 88 as the theoretical covering index (in the following it is assumed as 1, by normalization). We find only 11 pupils using exactly this numbers of tiles. On the other hand, 32 children choose to extend the paving beyond the sheet edges and use 96 tiles (constructing an 8×12 floor, they cover a hypothetical sheet of 20 cm × 30 cm). We obtain 56 protocols (more than 26.41%) which used from 88 to 96 tiles.

It is possible that the number of tiles the children glue on the sheet is determined by their attention span, by their manual coordination, by their commitment to the task, and by their interest in producing their own design, and also it can be related with age and teacher’s practice. Since it is simple to calculate it can be used as a rough indicator of all these aspects.

Children were free to use any of the tiles, so that individual artistic taste and choice of design are what determines the choice of tile. We recorded the number of tiles of each kind, in each protocol. We also introduced a diversity index, borrowed from biological research.

THE SCHOOLS.

The first author requested and obtained permission to conduct the experiments in the Kindergartens and Primary Schools of his home town, Viadana (MN) in the province of Mantova, Northern Italy [2]. Viadana is a small town with agriculture and artisan industry. There is a high number of immigrants from other Italian regions and from abroad, but there was no statistically significant difference among the protocols of Italian and non-Italian children, nevertheless, some specific protocols seem to be influenced by the familiar culture of these foreign pupils.

THE QUANTITATIVE RESULTS.

The first type of analysis of protocols is counting the number and type of tile each child used. It is remarkable that this analysis yields interesting information.

Covering index.

Table 1, below, shows the average covering index values. It is clear that the average covering index increases with the years of schooling. The presence of high scores for particular classes could be explained by pupils’ possible independent experience of a similar activity or by different teaching practices.

<table>
<thead>
<tr>
<th>School</th>
<th>no. pupils</th>
<th>Sample</th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kindergarten</td>
<td>97</td>
<td>53.5</td>
<td>47.9</td>
<td>59.8</td>
</tr>
<tr>
<td>1st grade Prim.</td>
<td>68</td>
<td>77.1</td>
<td>79.1</td>
<td>74.0</td>
</tr>
<tr>
<td>2nd grade Prim.</td>
<td>47</td>
<td>86.5</td>
<td>87.2</td>
<td>85.4</td>
</tr>
</tbody>
</table>

Table 1: Average Covering Index
In more details, Bedoli and San Pietro Kindergartens, in 1E and 2A of Primary School, the males’ average covering index is greater than the females’ one, in the remaining classes it is conversely. The following Table 2 shows the average covering index for each class.

**Diversity index.**

This index varies between 0 (every tile in the protocol of the same kind) and 2 (equal number of tiles of each kind in the protocol). In Biology, the good ecological ‘health’ of an environment is measured by a diversity index near to 2. In our work, we can consider the diversity index as a measure of the respect of the teachers for the propensities of pupils. The diversity index is calculated as follows. Let $N$ be the total number of tiles used. Let $n_i \ (i = 1, 2, 3, 4)$ be the relative number of flowery tiles (1), of branch tiles (2), of straight tiles (3), and of swallows tiles (4) used in protocols. First we calculate the rate of presence of each tile $\frac{n_i}{N}$, then the diversity index $D$, using the Shannon & Weaver, 1949 formula, $D = -\sum_{i=1}^{4} n_i \cdot \log_2 \left( \frac{n_i}{N} \right)$. Table 2 resumes, in diagrammatic style, the interesting relations between the two indexes we presented above. It is evident that in parallel classes, there is an inverse correlation between average covering index and average diversity index. We feel that this phenomenon could depend mainly on teacher practice. The different behaviour of these indexes among non parallel classes might be related with age and schooling.

<table>
<thead>
<tr>
<th></th>
<th>Covering index</th>
<th>Diversity index</th>
</tr>
</thead>
<tbody>
<tr>
<td>1E</td>
<td>2.00</td>
<td>1.50</td>
</tr>
<tr>
<td>2A</td>
<td>1.00</td>
<td>0.50</td>
</tr>
<tr>
<td>1A</td>
<td>0.50</td>
<td>1.00</td>
</tr>
<tr>
<td>1C</td>
<td>1.50</td>
<td>1.00</td>
</tr>
<tr>
<td>2C</td>
<td>0.50</td>
<td>1.50</td>
</tr>
<tr>
<td>Prim 1E</td>
<td>0.00</td>
<td>2.00</td>
</tr>
<tr>
<td>Prim 1A</td>
<td>0.00</td>
<td>1.50</td>
</tr>
<tr>
<td>Prim 2C</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Prim 2A</td>
<td>0.00</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 2: Average Covering and Diversity Indexes for classes

**Average use of tiles.**

Data in Table 3 show the number of tiles children used in paving the ‘floor’.

This Table is of immediate interest in that it shows a clear difference between males and females, which is evident even at Kindergarten and Primary School. In a sense the tiles have a gender connotation. The way girls and boys use the flowery tiles is
particularly striking. The ‘monopolization’ of the flowery tiles by girls lowers the diversity index. These facts are thence connected.

<table>
<thead>
<tr>
<th></th>
<th>Flowery rate of use</th>
<th>Branch rate of use</th>
<th>Straight rate of use</th>
<th>Swallows rate of use</th>
<th>Total n. of tiles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample</td>
<td>24.45</td>
<td>13.61</td>
<td>13.75</td>
<td>17.73</td>
<td>14,740</td>
</tr>
<tr>
<td>Sample Males</td>
<td>15.77</td>
<td>16.18</td>
<td>14.12</td>
<td>21.91</td>
<td>8,294</td>
</tr>
<tr>
<td>Sample Fem.</td>
<td>36.21</td>
<td>10.12</td>
<td>13.23</td>
<td>12.06</td>
<td>6,446</td>
</tr>
<tr>
<td>Kindergarten</td>
<td>28.91</td>
<td>15.50</td>
<td>15.76</td>
<td>22.85</td>
<td>9,548</td>
</tr>
<tr>
<td>K. Males</td>
<td>18.17</td>
<td>17.27</td>
<td>17.15</td>
<td>29.82</td>
<td>5,851</td>
</tr>
<tr>
<td>K. Females</td>
<td>46.25</td>
<td>12.66</td>
<td>13.50</td>
<td>11.61</td>
<td>3,697</td>
</tr>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; grade Prim</td>
<td>22.79</td>
<td>10.88</td>
<td>14.04</td>
<td>29.41</td>
<td>3,332</td>
</tr>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; grade Males</td>
<td>14.69</td>
<td>14.36</td>
<td>12.55</td>
<td>37.50</td>
<td>5,245</td>
</tr>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; grade Fem.</td>
<td>35.88</td>
<td>5.27</td>
<td>16.46</td>
<td>16.35</td>
<td>1,923</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; grade Prim</td>
<td>37.77</td>
<td>16.83</td>
<td>18.06</td>
<td>13.85</td>
<td>4,066</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; grade Mal.</td>
<td>23.21</td>
<td>21.48</td>
<td>23.83</td>
<td>18.69</td>
<td>2,529</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; grade Fem.</td>
<td>61.22</td>
<td>9.33</td>
<td>8.78</td>
<td>6.06</td>
<td>1,537</td>
</tr>
</tbody>
</table>

Table 3: Rates of tile use

We think that a quantitative investigation of this kind can give some interesting information to teachers. It seems us a tool which is enough simple to apply. Other more sophisticated analyses, e.g. feeble and rigid structures or the interpretation of gesture, require more competencies.

ANALYSYS OF THE PROTOCOLS AND A GEOMETRIC INTERPRETATIVE HYPOTHESIS OF THE QUALITATIVE RESULTS.

An analysis of protocols allows making an initial classification based on the criteria used by the pupils in the construction of the floor. We distinguish the following kinds of criteria:

0) *random*: pupils glue the tiles as they pick them up at random, without observing the drawing on them;
1) *taking account of the drawing on the tile*: on the straight tile the line is parallel to the edge; so the children tend glue the tile with the line ‘horizontal’ or ‘vertical’. Other tiles do not have a preferred direction, although tiles tend to be placed with their sides parallel to the edges of the paper as far as this is possible;
2) *influenced by and based on neighbouring tiles*: construction of a route, translation, symmetry; construction of a flower in the case of flowery;
3) *regular*: an iterative and regular tessellation;
4) *progressive conquest of regularity*: initially the pupils glue tiles at random and
subsequently choose regular tessellation;
5) project: pupils first ‘see’ a mental representation, and then proceed to the concrete manipulation.

In case 2) and 4) there are feeble structures, generally; in case 3) and 5) the structure is generally rigid, when the project is realized in a complete way or the regularity is observed in all the work. Sometimes pupils are unable to concretise their mental image obtaining feeble structures.

The drawings on the tiles are such that when the same type of tiles is placed next to one another, the lines fit together perfectly. This feature leads pupils to imagine drawing concrete things. For example, the *swallows* or *straight* tiles might give the idea of a road; the *branch* might give the idea of a scene from nature such as a thorny lawn, the *flowery* tile a garden. But pupils’ imagination is even more fertile than this; they ‘see’ for example a *chick* and a *cat* in the following arrangements of tiles.

![Figure 4: Chick and cat.](image)

So a further avenue for research might be the tessellations of *cats, chicks, bells* etc.

Geometry of tiles also influenced the children’s work. Their drawings are of three kinds from point of view of inherited symmetries:

1) *straight*: two reflections with orthogonal axis, the medians and consequently a central symmetry;

2) *branch and swallows*: two reflections with orthogonal axis, the diagonals and consequently a central symmetry;

3) *flowery*: one reflection on a diagonal line.

The *flowery* tile allows for different ‘interpretations’. The tiles can be glued so that the wider lines connect, or to compose whole flowers instead, without connection of wider lines. The arrangements with more symmetries followed fit both, see Appendix. The protocols show all these arrangements, often only locally; we think that it could be a result obtained by chance, but it can be a good occasion given to the teacher.

It may be the case that it is more mature pupils who create a greater number of basic motives (Budden, 1972). Some of the older Primary School children showed the presence of strong structures and pupils there tended to opt for the flowery tile. Girls are naturally more mature than boys and this may be the reason they chose the flowery tile (Arnheim, 2005). There was however a considerable increase in the use of the flowery tile in the second grade.
NOTES

1. Work done in the sphere of Local Research Unit into Mathematics Education, Parma University, Italy.

2. We would like thank the School Heads and the teachers of Bedoli, Carrobbio, Cogozzo, and San Pietro Kindergartens, and Classes 1A, 1C, 1E, 2A and 2C of Primary School, for permitting and collaborating with our work.

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APPENDIX